# Math 221 Fall 98 Computer Assignment 2

**Due on**

Please hand in the printout of a Maple worksheet as part of your solution to this problem. Working in a small group is encouraged.

## The Parallelogram Image Package

The routine pargrm_image_2d allows you to readily view the geometry of a linear transformation of $\mathbf{R^2}$. After starting Maple, you need to load the package with a line like:

**read ‘Pacific:othermaple:pargrm_image‘;**

where you will likely have to substitute for Pacific the name of your machine. The quotes here are backquotes in the upper left of your keyboard rather than apostrophes.

Suppose that vectors **v1** and **v2** are defined as in

**v1 := vector([1,0]);**
**v2 := vector([0,1]);**

and a matrix **A1** is defined by

**A1 := matrix(2,2,[.28, .96, .96, -.28]);**

Then the Maple call

**gr1 := pargrm_image_2d([v1,v2],A1);**
**display(gr1, scaling=constrained):**

will display two parallelograms with vertices at the origin. One parallelogram $P1$ will have edges **v1** and **v2**, and be colored blue. The other will be the image under **A1** of this parallelogram, and so will have edges **A**(**v1**), and **A**(**v2**). This image is colored red.

At places where the original and image parallelograms overlap, the color will be blue, but you will still be able to see the edges **A**(**v1**) and **A**(**v2**).

The line *scaling=constrained* above tells Maple that you'd like the $x$ and $y$ axis scales to be the same. This is essential if you are trying to read off information like distance and angles from a picture !

There is also a routine pargrm_image_3d using 3 dimensional vectors, and a $3 \times 3$ matrix **A** to draw the images under **A** of a parallelepiped with edges **v1**, **v2**, and **v3**. A typical call would be

**pargrm_image_3d([v1,v2,v3],A);**

Samples using the parallelogram image package are located in the file *:Maple V Release 4:Math 221:Parallelogram Image* on each Macintosh in the Lab.

## Some Matrices and Parallelograms

In this problem we'll want to consider the vectors:

$$\mathbf{v1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{v3} = \begin{pmatrix} .6 \\ .8 \end{pmatrix}, \mathbf{v4} = \begin{pmatrix} -.8 \\ .6 \end{pmatrix}, \mathbf{v5} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and

$$\mathbf{v6} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

We will also consider the parallelograms:

- **P1** with edges **v1** and **v2**.
- **P2** with edges **v3** and **v4**.
- **P3** with edges **v5** and **v6**.
- **P4** with edges **v1** and **v6**.

And the matrices

$$\mathbf{A1} = \begin{pmatrix} .28 & .96 \\ .96 & -.28 \end{pmatrix}, \mathbf{A2} = \begin{pmatrix} .28 & .96 \\ -.96 & .28 \end{pmatrix}, \mathbf{A3} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix},$$

as well as

$$\mathbf{A4} = \begin{pmatrix} .04 & .72 \\ -1.28 & 1.96 \end{pmatrix}, \mathbf{A5} = \begin{pmatrix} 2.36 & .48 \\ .48 & 2.64 \end{pmatrix}, \mathbf{A6} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix},$$

and

$$\mathbf{A7} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$

It is probably a good idea to start out by entering these vectors and matrices in your Maple session. The file *:Maple V Release 4:Math 221:Spring 98 #2 vectors and matrices* contains these already typed in. Remember you have to actually execute the lines in your current session for the assignments to take effect.

## Questions

As you do the following questions, we'd like you to look at a fair number of results of

**gr1 := pargrm_image_2d([vi,vj],Ak);**
**display(gr1, scaling=constrained);**

but don't print them all. Also note that once you have two lines like the above in your Maple session, you can just edit the indices $i$,$j$,$k$ and then hit *ENTER* again to see the next set of pictures.

In these questions, some parts involve looking at a computer result, while others involve a mathematical argument. We'll use the symbol $^\dagger$ to indicate questions whose answers are primarily of a computer nature and $^*$ for questions requesting other arguments. Some parts of the questions, are providing defintions and explanations; you don't need to answer those of course!

**a)** A square matrix $\mathbf{A}$ with the property that $\mathbf{A^tA}$ is the identity is called *orthogonal.* Show$^\dagger$ that $\mathbf{A1}$ and $\mathbf{A2}$ meet this definition of orthogonality. (The importance of the definition is that the associated linear transformations preserve distances and angles.)

**b)** Look$^\dagger$ at the result of transformation $\mathbf{A1}$ on the four parallelograms above. A *reflection* of $\mathbf{R^2}$ is an orthogonal transformation $\mathbf{T}$ with the further property that it leaves one nonzero vector $\mathbf{u1}$ fixed (i.e. $\mathbf{T(u1)} = \mathbf{u1}$) and reverses another vector $\mathbf{u2}$ (i.e. $\mathbf{T(u2)} = -\mathbf{u2}$). The vector $\mathbf{u1}$ is called the *axis* of the reflection. Print$^\dagger$ out one picture suggesting what the axis of $\mathbf{A1}$ is likely to be. Then show$^*$ algebraically that vectors $\mathbf{u1}$ and $\mathbf{u2}$ as above exist for $\mathbf{A1}$.

**c)** Look$^\dagger$ at the result of transformation $\mathbf{A2}$ on the four parallelograms above. Can you see why$^\dagger$ the word *rotation* is used to describe the action of $A2$ ? (To explain this, relate a picture to the ordinary English meaning of the word.) Show$^*$ that $\mathbf{A2}$ has no nonzero fixed vectors, i.e. $\mathbf{A2(u)} \neq \mathbf{u}$ for all vectors $\mathbf{u} \neq 0$.

**d)** It can be shown that all rotations are of the form

$$\mathbf{R} = \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix},$$

for some angle $\theta$. Show$^*$ that the matrices $\mathbf{A2}$ and $\mathbf{A1^2}$ are both rotations. (It is in fact true that the composition of two reflections is always a rotation.)

**e)** Look$^\dagger$ at the result of transformation $\mathbf{A3}$ on the four parallelograms above. A *shear transformation* of $\mathbf{R^2}$ is a linear transformation with the properties:

1. It leaves one nonzero vector $\mathbf{u1}$ fixed (i.e. $\mathbf{T(u1)} = \mathbf{u1}$). The vector $\mathbf{u1}$ is the *axis* of the shear.
2. For any vector $\mathbf{u2}$, the difference between $\mathbf{T(u2)}$ and $\mathbf{u2}$ is parallel to the axis $\mathbf{u1}$.

By looking at the results of the transformation **A3** on the four parallelograms above, find† the axis of the shear **A3**. Also, show algebraically* that **A3** meets the definition above of a shear transformation.

**f)** Look† at the result of linear transformation **A4** on parallelogram $P2$. Explain using the definition above* why this picture suggests **A4** is a shear. What is its axis († or *)?

**g)** Look† at the result of transformation **A6** on the four parallelograms above. A *dilation* is a linear transformation preserving angles and directions, but not necessarily distances. Show* for any two vectors **u1** and **u2**, the angle between them is the same as the angle between their images **A6**(**u1**) and **A6**(**u2**)

**h)** Look† at the result of transformation **A5** on parallelogram $P2$. Give† a description of what you see in terms of change of scale of the sides of $P2$. Does† this change of scale interpretation appear to hold for any of the other parallelograms ?

**i)** Print† out the result of transformation **A7** on one of these parallelograms, and explain* the result.

One reason for talking about shears, dilations, and orthogonal transformations is that any linear transformation of $\mathbf{R^2}$ can be expressed as a composition of one each of these together with one more basic transformation, a *strain*. A typical strain would be given by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

This also generalizes to higher dimensions.