

Due in Recitation on Thursday September 17

Please hand in the printout of a Maple worksheet as your solution to this problem. Working in a small group is encouraged.

Maple is available in the Mathlab, Stimson 206 which is open about 70 hours per week. (This includes til 10PM on Sundays thru Thursdays, and from 9AM Monday thru Friday.)

You should find the handouts on Maple in General and the Row Operations package helpful on this. If you run into Maple syntax difficulty, don't hesitate in turning to the consultants in the Lab, all of whom are experienced in Maple.

A Population Problem

One application of linear algebra is to the evolution of population distributions over time.

For example, consider the following data about regional population distribution in the United States. If the population redistribution seen in this table

	Region	1980 Population	1980 fraction	1990 Population	1990 fraction
1.	Northeast	49.1 M	.2167	50.8 M	.2043
2.	Midwest	58.9 M	.2599	59.7 M	.2400
3.	West	43.2 M	.1906	52.8 M	.2123
4.	South	75.4 M	.3327	85.4 M	.3434

Figure 1: Regional Population Shifts

were to continue, what might that mean for the future? In this problem, you'll look at a Markov chain model to explore this. First we review how such a model works:

Markov Chain Models

Suppose a population consists of k disjoint sub-populations labeled by the integers $1 \dots k$. Let p_i be the fraction of the population in sub-population i . Form the vector $\vec{S} \in R^k$ called the state vector of the system:

$$\vec{S} = \begin{pmatrix} p_1 \\ p_2 \\ \dots \\ p_k \end{pmatrix}$$

The sum $\sum_{i=1}^k p_i$ of the population fractions is assumed to be 1.

The state vector \vec{S} will depend on time t . In a discrete model, we might measure the population fractions p_i at times $\{t_1, t_2, \dots, t_n, \dots\}$. Use $\vec{S}[t_n]$ and $p_i[t_n]$ respectively to denote values at the end of the n 'th time period.

In a Markov chain model one assumes that there are transition probabilities T_{ij} representing the proportion of the population in sub-population j during time period n which enter sub-population i for time period $n + 1$. These transition probabilities are independent of time. Let T be the $k \times k$ matrix whose i, j entry is T_{ij} .

We also assume that the entire population in time period $n + 1$ comes by this transition process from the population during time period n . Thus

$$p_i(t_{n+1}) = T_{i1}p_1(t_n) + T_{i2}p_2(t_n) + \cdots + T_{ik}p_k(t_n) +$$

This is just the formula for the i 'th entry of a matrix product of T and the n 'th state vector $\vec{S}[t_n]$. Thus in matrix form, $\vec{S}[t_{n+1}] = T \circ \vec{S}[t_n]$.

More generally, we can keep track of the change in population over r time periods by computing the r 'th power of T . Specifically $\vec{S}[t_{n+r}] = T^r \circ \vec{S}[t_n]$.

The sum of the entries in each column (e.g. the j 'th) of the transition matrix T should be 1 because the entries of that column reflect all the possible destinations at time t_{n+1} for the people in region j at time t_n .

Analyzing the Population Changes

The state data for 1980 in the table of regional population distribution above could be entered in Maple using the commands:

```
with(linalg);
S[80] := vector([.2167, .2599, .1906, .3327]);
```

You can enter the data $S[90]$ for 1990 similarly.

Let T be the 4×4 matrix

$$\begin{pmatrix} .8166 & .085 & .01 & .01 \\ .01 & .895 & .01 & .01 \\ .099 & .01 & .97 & .01 \\ .0744 & .01 & .01 & .97 \end{pmatrix}$$

This problem will start out by testing the appropriateness of T as a possible transition matrix describing regional population shifts over a decade.

1. By multiplying T on the left by the row vector $[1, 1, 1, 1]$, show that the sums of the columns of T are close to 1. (Explain why multiplying by this row vector is calculating the column sum.)
2. Show that T approximately accounts for the change from 1980 to 1990, i.e. $T \circ S[80]$ is approximately $S[90]$.
3. Compute $T^{50} \circ S[80]$, $T^{100} \circ S[80]$, and $T^{200} \circ S[80]$, and compare these results. What does the comparison suggest ?
4. Use the row operations package to approximately solve the system $T \circ w = w$. Explain why a state vector w satisfying this equation would represent an unchanging (*steady-state*) population distribution for this model.

Maple Comments:

- Samples using the row operations package are located in the file *:Maple V Release 4:Math 221: Row Operations Examples* on each Macintosh in the Lab.
- To solve this system, you need only solve $A \circ w = 0$ where A is the matrix $T - Id$. The Maple commands

```

Id := diag(1, 1, 1, 1);
A := T - Id;

```

will generate these matrices.

- A single use of the row operations package to solve $A \circ x = b$ will include a *read* statement to load the package, a definition of A and b , a “*start_ge(A,b);*” call, and then a sequence of row operations (*ar*, *mr* and *sr*) as well as a back-substitution (*bs()*). For decimals, one can use (*rounded_bs(k)*) to round to k decimal places, thereby rounding small entries (from roundoff error) to zero.
- Note also, that in using the row operations package, you don’t have to do arithmetic — you can issue commands like:

```

ar(1,2, .7865/.2345);

```

- By approximately solve, we mean just work to 4 digit accuracy in your row operations, and assume that naturally arising terms near 0 differ from 0 only because of roundoff error.
- There is a mathematically delicate issue associated with rounding here. The square system $(T - Id)w = 0$ is solvable nontrivially only if $T - Id$ is a singular matrix. Doing row operations *with floating point* may change this into a matrix that is non-singular although “almost” singular. A rigorous theory of when and how to replace an almost singular matrix by a singular one is somewhat difficult. Here we encourage you to just informally assume nearly zero entries are really zero. **But this will only work if you do not unnaturally scale the entries.** For example if you have a row every entry of which is 0 except for 0.00001, then rounding to 4 decimal places will round all entries to 0. But multiplying the row by 10^5 to make this nonzero entry 1 would defeat the purpose of rounding.

5. Compare your steady state answer to the result of $T^{100} \circ S[80]$ above. In converting your solution to one the sum of whose components is 1, you may find it helpful to use the Maple commands

```

v_sum := add(v[i],i=1..4);

```

if \mathbf{v} is a vector or

```

v_sum := add(v[i,1],i=1..4);

```

if \mathbf{v} is a 4×1 matrix to add up the 4 components of \mathbf{v} and

```

mult := u -> u/v_sum;
map(mult,v);

```

The latter define a helper function multiplying any number by $1/v_sum$, and then apply that function to each entry of v .

The transition matrix T is just one Markov chain model consistent with the data. You might find it interesting to think about other possibilities.

The fact that **3** and **4** agree can be shown to hold in general for Markov matrices as long as some power has all its entries strictly positive. But it’s not obvious