Pythagorean Triplets

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We seek to identify all positive integers a, b, and c so that $a^2+b^2 = c^2$. A triple (a,b,c) satisfying this equation is called a *pythagorean triplet.* Since any multiple of a pythagorean triplet is again one, we only seek *primitive* ones where a, b, and c have no common factor.

By setting $x = \frac{a}{c}$ and $y = \frac{b}{c}$, we can first seek rational numbers x and y so that $x^2 + y^2 = 1$.

Now, if (x, y) is such a pair of rational numbers, the line from (x, y) to $(-1, 0)$ will have rational slope - call it t. Along this line, $y = t(x + 1)$, so $x^2 + t^2(x + 1)^2 = 1$ telling us that

$$(1 + t^2)x^2 + 2t^2x = 1 - t^2$$

or

$$x^2 + \frac{2t^2}{1+t^2}x = \frac{1-t^2}{1+t^2}$$

Complete the square on the above equation giving

$$\begin{aligned}(x + \frac{t^2}{1+t^2})^2 &= \frac{1-t^2}{1+t^2} + \frac{t^4}{(1+t^2)^2} \\ &= \frac{1}{(1+t^2)^2}\end{aligned}$$

Taking square roots gives

$$x = \frac{1-t^2}{1+t^2}.$$

Write $t = \frac{u}{v}$ with u, v positive integers having no common factor. This gives

$$\begin{aligned} x &= \frac{v^2-u^2}{v^2+u^2} \\ y &= \frac{2uv}{v^2+u^2} \end{aligned}$$

Using $x = \frac{a}{c}$ and $y = \frac{b}{c}$ we see that

$$\begin{aligned} a &= k(v^2-u^2) \\ b &= k(2uv) \\ c &= k(v^2+u^2) \end{aligned}$$

for some rational number k.

By thinking about the highest power p^r of an odd prime appearing in the denominator of k (written in lowest terms), one sees that p^r divides uv, u^2+v^2, $-u^2+v^2$, $2u^2$, and $2v^2$. This means $p^{\frac{r}{2}}$ divides both u and v contradicting u and v having no common factor, UNLESS $r = 0$. Similarly by noting exactly one of a and b can be odd, and choosing b to be the even number, we can eliminate powers of 2 in the denominator of k. This shows that k is a whole number and we obtain:

Theorem: The pythagorean triplets with b even are exactly the triples of the form

$$\begin{aligned} a &= v^2-u^2 \\ b &= 2uv \\ c &= v^2+u^2 \end{aligned}$$

where u and v are positive whole numbers with no common factor.