The curve is
a) an ellipse if $4AC - B^2 > 0$,
b) a hyperbola if $4AC - B^2 < 0$,
c) a parabola if $4AC - B^2 = 0$.
The last equation can be rewritten as

$$(1 - c_2^2)x^2 + (1 - c_1^2)y^2 - 2c_1c_2xy - 2dc_1y - 2dc_2x - d^2 = 0.$$

So $A = 1 - c_2^2$, $B = -2c_1c_2$ and $C = 1 - c_1^2$. Thus to get a parabola, we must have

$$4c_1^2c_2^2 = 4(1 - c_1^2)(1 - c_2^2).$$

Simplifying we get the relation $c_1^2 + c_2^2 = 1$. Substitute the expressions for c_1 and c_2 in terms of v_1 and v_2. You get a relation between v_1 and v_2 which insures that the path is a parabola. You can plug in various values for v_1 and solve for v_2.

4. For part 4, you plug $x = -1000$ and $y = -1000$ in the last equation, $(1 - c_2^2)x^2 + (1 - c_1^2)y^2 + \cdots = 0$. This gives a relation between c_1 and c_2. Expressing c_1, c_2 in terms of v_1, v_2, you get a equation that v_1, v_2 have to satisfy. You can then find particular solutions by plugging in specific values for v_1 and solving for v_2 in the equation.

Homework

Assume that $G = 1$ and that $M = 10^6$. Suppose that $\mathbf{r}_0 = (0, 10^3)$ and $\mathbf{v}_0 = (v_1, v_2)$.

1. Find the value of h.

2. Find the constants c_1, c_2.

3. For what initial velocities will the path be a circle? For what initial velocities wil the path be a parabola?

4. Find a velocity so that the path goes through $(-1000, -1000)$.

5. Take your answers to the Math Lab and try them out on the program *Planets* which you find in the folder Maple/Math213. Ask if you cannot find the file. The initial conditions should be set, but double check them.

Some Hints

1. The formula for h in terms of the initial position and initial velocity is on page 1, $h = r_2 v_1 - r_1 v_2$.

2. $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$. The velocity is really $(v_1, v_2, 0)$ but since the motion is in the xy–plane, we suppress the last zero (same for $\mathbf{r}$). This is particularly the case for the formulas involving cross products. For example in formula (9),

$$\mathbf{v}_0 \times \mathbf{k} = det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = v_2 \mathbf{i} - v_1 \mathbf{j}.$$

The vector $\frac{1}{r_0}\mathbf{r}_0$ is the unit vector in the direction of $\mathbf{r}_0$ in this case $(0, 1)$. The correct way to write it would be

$$\mathbf{c}_0 = (c_1, c_2, 0) = (\frac{r_1}{(r_1^2 + r_2^2)^{1/2}}, \frac{r_2}{(r_1^2 + r_2^2)^{1/2}}, 0) - \frac{h}{GM}(v_2, -v_1, 0).$$

Since the third coordinate is always zero, it is suppressed. You have to plug in the values for h and r_1, r_2 and find expressions for c_1 and c_2.

3. Now for the hard part:

Consider the general equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{14}$$

This is called a conic. It is either an ellipse, a hyperbola or a parabola. To decide which it is, complete the square in the quadratic term:

$$\begin{aligned} &Ax^2 + Bxy + Cy^2 = A(x^2 + \frac{B}{A}xy + \frac{C}{A}y^2) = \\ &A[(x + \frac{B}{2A}y)^2 + \frac{C}{A}y^2 - \frac{B^2}{4A^2}y^2] = A[(x + \frac{B}{2A}y)^2 + \frac{4AC - B^2}{4A^2}y^2]. \end{aligned} \tag{15}$$

The constant $\mathbf{c}_0 = (c_1, c_2)$ can be computed from the initial conditions,

$$\mathbf{c}_0 = \frac{1}{r_0}\mathbf{r}_0 - \frac{h}{GM}\mathbf{v}_0 \times \mathbf{k}. \tag{9}$$

Let us write equation (8) in Cartesian coordinates:

$$\frac{x}{(x^2+y^2)^{1/2}}\mathbf{i} + \frac{y}{(x^2+y^2)^{1/2}}\mathbf{j} = (\frac{h}{GM}y' + c_2)\mathbf{i} + (-\frac{h}{GM}x' + c_1)\mathbf{j}. \tag{10}$$

(I used the fact that $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$ and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ to derive $\mathbf{v} \times \mathbf{k} = (x'\mathbf{i} + y'\mathbf{j}) \times \mathbf{k} = y'\mathbf{i} - x'\mathbf{j}$.) Thus

$$-\frac{h}{GM}x' + c_1 = \frac{y}{(x^2+y^2)^{1/2}}, \qquad \frac{h}{GM}y' + c_2 = \frac{x}{(x^2+y^2)^{1/2}}. \tag{11}$$

We can rewrite the relation $\mathbf{r} \times \mathbf{v} = h\mathbf{k}$ as $x'y - y'x = h$. Thus if we multiply x' by y and y' by x and add the two equations in (11):

$$\frac{h^2}{GM} + c_1 y + c_2 x = \sqrt{x^2+y^2}. \tag{12}$$

(I simplified some arithmetic on the right hand side!). Squaring we find

$$x^2 + y^2 = (d + c_2 x + c_1 y)^2 = c_1^2 y^2 + c_2^2 x^2 + 2c_1 c_2 xy + 2dc_1 y + 2dc_2 x + d^2. \tag{13}$$

This is a conic section (quadratic equation) as promised; I set $d = \frac{h^2}{GM}$.

These formulas are helpful for computing the derivatives in t. Note the use of the chain rule:

$$\frac{d\mathbf{u}_r}{dt} = \frac{d\mathbf{u}_r}{d\theta}\frac{d\theta}{dt} = \frac{d\theta}{dt}\mathbf{u}_\theta, \qquad \frac{d\mathbf{u}_\theta}{dt} = \frac{d\mathbf{u}_\theta}{d\theta}\frac{d\theta}{dt} = -\frac{d\theta}{dt}\mathbf{u}_\theta.$$

Then velocity and acceleration in terms of $\mathbf{u}_r,\ \mathbf{u}_\theta$ are

$$\begin{aligned} \mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\mathbf{u}_r}{dt} = \frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta, \\ \mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d^2r}{dt^2}\mathbf{u}_r + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + \frac{dr}{dt}\frac{d\theta}{dt}\mathbf{u}_\theta + r\frac{d^2\theta}{dt^2}\mathbf{u}_\theta - r(\frac{d\theta}{dt})^2\mathbf{u}_r = \\ & [\frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2]\mathbf{u}_r + [2\frac{dr}{dt}\frac{d\theta}{dt} + \frac{d^2\theta}{dt^2}]\mathbf{u}_\theta. \end{aligned} \tag{3}$$

Since $\mathbf{a}$ is parallel to $\mathbf{u}_r$, the coefficient of $\mathbf{u}_\theta$ must be zero:

$$\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = 0.$$

But note that

$$\frac{d}{dt}(r^2\frac{d\theta}{dt}) = 2r\frac{dr}{d\theta} + r^2\frac{d^2\theta}{dt^2} = r[\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}] = 0.$$

Thus $r^2\frac{d\theta}{dt}$ is a constant in t. Recall that

$$h\mathbf{k} = \mathbf{r}\times\mathbf{v} = r\mathbf{u}_r \times (\frac{dr}{dt}\mathbf{u}_r + r\frac{d\theta}{dt}\mathbf{u}_\theta) = \mathbf{u}_r \times \frac{dr}{dt}\mathbf{u}_r + r\mathbf{u}_r \times r\frac{d\theta}{dt}\mathbf{u}_\theta = r^2\frac{d\theta}{dt}\mathbf{k}$$

because $\mathbf{u}_r \times \mathbf{u}_r = \mathbf{0}$. Thus

$$r^2\frac{d\theta}{dt} = h. \tag{4}$$

We can equate coefficients of $\mathbf{u}_r$ in (2) and (3) as well:

$$\frac{d^2r}{dt^2} - r(\frac{d\theta}{dt})^2 = -\frac{GM}{r^2}. \tag{5}$$

We can substitute (4) into (5) to get rid of $\frac{d\theta}{dt}$:

$$\frac{d^2r}{dt^2} - r(\frac{h}{r^2})^2 = -\frac{GM}{r^2}, \qquad \frac{d^2r}{dt^2} - \frac{h^2}{r^3} = -\frac{GM}{r^2}, \qquad \frac{d^2r}{dt^2} = \frac{h^2}{r^3} - \frac{GM}{r^2}. \tag{6}$$

We will deal with such equation later in the semester. For now let us observe that we can use (4) and (2) in the following manner:

$$\frac{d\mathbf{u}_r}{dt} = \frac{d\theta}{dt}\mathbf{u}_\theta = -\frac{d\theta}{dt}\mathbf{u}_r\times\mathbf{k} = \frac{d\theta}{dt}\frac{r^2}{GM}\mathbf{a}\times\mathbf{k} = \frac{h}{GM}\mathbf{a}\times\mathbf{k} = \frac{d}{dt}[\frac{h}{GM}\mathbf{v}\times\mathbf{k}]. \tag{7}$$

Thus we can integrate and get

$$\mathbf{u}_r = \frac{h}{GM}\mathbf{v}\times\mathbf{k} + \mathbf{c}_0. \tag{8}$$

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Kepler's Laws

In these notes we want to derive the formula for the path of an object as a function of its initial position and initial velocity. We are assuming that there are two bodies, one of mass m at position $\mathbf{r}_0$ with initial velocity $\mathbf{v}_0$ at time $t = 0$, and the other of mass M. The force acting on m is given by Newton's law,

$$\text{(1)} \qquad \mathbf{F} = -\frac{GMm}{r^3}\mathbf{r}, \qquad \mathbf{F} = m\mathbf{a} \text{ or substituting, } \mathbf{a} = -\frac{GM}{r^3}\mathbf{r}.$$

You can imagine the sun with mass M being positioned at the origin $(0,0,0)$. You may also assume that it is not moving, and we ignore the existence of any planets as their mass is much smaller. An asteroid or comet or rocket is positioned at $\mathbf{r}_0 = (r_1, r_2, r_3)$ at time $t = 0$. It is given an initial velocity $\mathbf{v}_0 = (v_1, v_2, v_3)$. The question is, *what path will it take?* The derivation is in the text, here we give more details and write it out explicitly so that we might be able to test with a computer simulation.

We first discuss the part that the motion is in a plane done in class. Formula (1) says that $\mathbf{r}$ and $\mathbf{a}$ are parallel, so $\mathbf{r} \times \mathbf{a} = 0$. But then we can write

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \frac{d\mathbf{r}}{dt} \times \mathbf{v} + \mathbf{r} \times \frac{d\mathbf{v}}{dt} = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} = \mathbf{r} \times \mathbf{a} = 0.$$

This implies that $\mathbf{r} \times \mathbf{v}$ is constant in t; we denote this vector by $\mathbf{h}$. We may as well assume that this vector is in the $\mathbf{k}$ direction. We write it as $\mathbf{h} = h\mathbf{k}$. If $h = 0$, then $\mathbf{r}$ is parallel to $\mathbf{v}$ at all times ($\mathbf{r} \times \mathbf{v} = \mathbf{0}$) and we saw in class that the motion has to be along a line. We assume that $h \neq 0$. Then we may as well write $\mathbf{r}_0 = (r-1, r-2)$ and $\mathbf{v}_0 = (v_1, v_2)$ as $r_3 = 0$ and $v_3 = 0$. The value of h can be computed from the initial conditions:

$$\mathbf{h} = h\mathbf{k} = \mathbf{r}_0 \times v_0 = (r_1\mathbf{i} + r_2\mathbf{j}) \times (v_1\mathbf{i} + v_2\mathbf{j}) = r_1v_1\mathbf{i} \times \mathbf{i} + r_1v_2\mathbf{i} \times \mathbf{j} + r_2v_1\mathbf{j} \times \mathbf{i} + r_2v_2\mathbf{j} \times \mathbf{j} = (r_1v_2 - r_2v_1)\mathbf{k}.$$

So $h = r_1v_2 - r_2v_1$.

In the plane xy we write everything in polar coordinates. We may as well assume it is in the xy–plane (this amounts to choosing a coordinate frame). Recall polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, where $r = \sqrt{x^2 + y^2}$. We introduce some new vectors:

$$\mathbf{u}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}, \qquad \mathbf{u}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}.$$

These are both unit vectors and you can check that they are orthogonal. In fact, the following relations hold:

$$\mathbf{u}_r \times \mathbf{u}_\theta = \mathbf{k}, \quad \mathbf{u}_\theta \times \mathbf{k} = \mathbf{u}_r, \quad \mathbf{k} \times \mathbf{u}_r = \mathbf{u}_\theta.$$

So these vectors are just like $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. We try to write everything in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$. The equations (1) become

$$\text{(2)} \qquad \mathbf{F} = -\frac{GMm}{r^3}r\mathbf{u}_r = -\frac{GMm}{r^2}\mathbf{u}_r, \qquad \mathbf{a} = \frac{-GM}{r^2}\mathbf{u}_r.$$

We now compute velocity and acceleration in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$. First we observe that

$$\frac{d\mathbf{u}_r}{d\theta} = \mathbf{u}_\theta, \qquad \frac{d\mathbf{u}_\theta}{d\theta} = -\mathbf{u}_r.$$

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